Nemytskij operator, Rådström embedding and set-valued functions

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Abstract. We give several results concerning a Nemytskij operator, generated by a set-valued functions. We consider two function spaces, namely the C^1 and AC spaces of continuously differentiable, resp., absolutely continuous, set-valued functions. We prove that the situation in which the Nemytskij operator is Lipschitzian continuous is characterized by a specific form of a function which generates the operator.

Keywords: Nemytskij operator, C^1 space, AC space, set-valued functions, Jensen equation.

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1. Introduction

All linear spaces considered in this article are assumed to be real. In the following we shall write I instead of $[0, 1]$.

In 1982 J. Matkowski showed (cf. [15]), that a Nemytskij operator N (which is defined by the formula $\phi \mapsto N(\phi) := g(\cdot, \phi(\cdot))$, where g is a given function) maps the function space $\mathrm{Lip}(I, \mathbb{R})$ into itself and is Lipschitzian with respect to the Lipschitzian norm if and only if its generator is of the form

$$g(x, y) = a(x)y + b(x), \quad x \in I, \quad y \in \mathbb{R},$$

for some $a, b \in \mathrm{Lip}(I, \mathbb{R})$. This result was extended to a lot of spaces by J. Matkowski and others (cf. e.g. [12, 13, 16]), in particular to the spaces $C^k(I, \mathbb{R})$ and $AC(I, \mathbb{R})$ of all k-times continuously differentiable, resp., absolutely continuous, functions $\phi\colon I \to \mathbb{R}$ (cf. [17]). Recently Matkowski has shown (cf. [18]) that if we only assume that the operator N is uniformly continuous, then the generator g is of the form above.

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Set-valued versions of Matkowski's results were investigated for instance in papers [5, 6, 7, 9, 10, 11, 19, 23, 24] and [25]. The main goal of this paper is to examine a set-valued analogue of Matkowski's result in the cases of the C^1 and AC spaces.

If $(Z, ||\cdot||_Z)$ is a normed space, then by $cc(Z)$ we denote the space of all non-empty, compact and convex subsets of Z. If A and B are subsets of Z, then we define $A+B := \{a+b : a \in A, b \in B\}$ and $\alpha A := \{\alpha a : a \in A\}$, where $\alpha \in \mathbb{R}$. Moreover, if $\alpha, \beta \in \mathbb{R}$ and $A, B \in cc(Z)$, then

$$\alpha(A+B) = \alpha A + \alpha B, \qquad \alpha(\beta A) = (\alpha\beta)A, \quad 1A = A,$$

and if $\alpha, \beta \geqslant 0$, then $(\alpha+\beta)A = \alpha A + \beta A$.

Let d denote the Hausdorff metric on the space $cc(Z)$, defined by the fromula

$$d(A,B) := \inf\{t \geqslant 0 : A \subseteq B + tS, \quad B \subseteq A + tS\},$$

where S is a closed unit ball in the space Z.

If $A \in cc(Z)$, then let us define $||A||_{cc(Z)}$ as follows:

$$||A||_{cc(Z)} := \sup\{||z||_Z : z \in A\}. \tag{1}$$

Moreover, if $\mathbf{C}$ is a non-empty subset of a real linear space, then we shall say that $\mathbf{C}$ is a *convex cone*, if it satisfies the following two conditions: $\mathbf{C}+\mathbf{C} \subseteq \mathbf{C}$ and $\lambda\mathbf{C} \subseteq \mathbf{C}$ for all $\lambda \geqslant 0$.

Lemma 1.1 ([21], Lemma 2)**.** *Let Z be a normed space. If A, B and C are non-empty, compact and convex subsets of Z, then $d(A+B, A+C) = d(B,C)$.*

Lemma 1.2 ([20], Theorem 5.6, p. 64)**.** *Let Y be a vector space and let Z be a Hausdorff topological vector space. Moreover, let $\mathbf{C}$ be a convex cone in Y. A set-valued function F defined on $\mathbf{C}$, with non-empty and compact values in Z, satisfies the Jensen equation*

$$F\Big(\frac{1}{2}(y_1+y_2)\Big) = \frac{1}{2}\big(F(y_1)+F(y_2)\big), \quad y_1, y_2 \in \mathbf{C},$$

if and only if there exist an additive set-valued function A, defined on $\mathbf{C}$ with non-empty, compact and convex values in Z and a non-empty, compact and convex subset B of Z such that $F(y) = A(y) + B$, $y \in \mathbf{C}$.

Theorem 1.3 ([21])**.** *For every normed linear space Z there exists a normed linear space $(V_Z, ||\cdot||_{V_Z})$ and an isometric embedding $\pi\colon cc(Z) \to V_Z$, where $cc(Z)$ is endowed with the Haussdorf distance d, for which $\pi(cc(Z))$ is a convex cone in V_Z and the conditions*

$$\begin{gathered} \pi(cc(Z)) - \pi(cc(Z)) = V_Z \\ \pi(A+B) = \pi(A) + \pi(B) \\ \pi(\alpha A) = \alpha\pi(A) \end{gathered} \tag{2}$$

are satisfied for $A, B \in cc(Z), \alpha \geqslant 0$. Moreover, V_Z is essentialy unique, i.e. if V_Z^1 and V_Z^2 are normed linear spaces and $\pi^1\colon cc(Z) \to V_Z^1$, $\pi^2\colon cc(Z) \to V_Z^2$ are embeddings which satisfy the above conditions, then there exists exactly one isometric isomorphism $T\colon V_Z^1 \to V_Z^2$ for which $T \circ \pi^1 = \pi^2$.

If E, E' are arbitrary non-empty sets, by $\mathcal{F}(E, E')$ we denote the set of all functions $f\colon E \to E'$. Every function $g\colon I \times E \to E'$ generates the so-called *Nemytskij operator* $N\colon \mathcal{F}(I, E) \to \mathcal{F}(I, E')$, defined by the formula

$$(N\phi)(x) := g(x, \phi(x)), \quad \phi \in \mathcal{F}(I, E), \quad x \in I. \tag{3}$$

For a function $A\colon I \times \mathbf{C} \to cc(Z)$ we shall write $A_y = A(\cdot, y), A^x = A(x, \cdot), x \in I, y \in \mathbf{C}$. Thus $A_y\colon I \to cc(Z)$ for $y \in \mathbf{C}$ and $A^x\colon \mathbf{C} \to cc(Z)$ for $x \in I$.

Let Y, Z be normed linear spaces, and let $\mathbf{C}$ be a convex cone in Y ($\mathbf{C}$ is endowed with the metric induced from Y). Consider the set

$$\mathcal{L}(\mathbf{C}, cc(Z)) := \{A\colon \mathbf{C} \to cc(Z) : A \text{ is additive and continuous}\}.$$

The formula

$$d_{\mathcal{L}}(A, B) := \sup_{y \in \mathbf{C} \setminus \{0\}} \frac{d(A(y), B(y))}{||y||_Y} \tag{4}$$

defines a metric in $\mathcal{L}(\mathbf{C}, cc(Z))$ (cf. [23] and [25]). Next the functional

$$||A||_{\mathcal{L}} := \sup_{y \in \mathbf{C} \setminus \{0\}} \frac{||A(y)||_{cc(Z)}}{||y||_Y}, \qquad A \in \mathcal{L}(\mathbf{C}, cc(Z)), \tag{5}$$

is not a norm, since $\mathcal{L}(\mathbf{C}, cc(Z))$ with addition and multiplication by real scalars, defined in the usual way, is not a vector space (except the case that Z is a single-point space).

Lemma 1.4 ([22], Lemma 5)**.** *Let Y and Z be normed linear spaces and let $\mathbf{C}$ be a convex cone in Y with nonempty interior. Then there exists a positive constant M_0 such that for every additive and continuous set-valued function $F\colon \mathbf{C} \to cc(Z)$ (in particular, for the functions the values of which are singletons) the inequality*

$$d(F(y_1), F(y_2)) \leqslant M_0 ||F||_{\mathcal{L}} ||y_1 - y_2||_Y$$

holds.

By $L(Y, V)$ we denote the normed space of all continuous linear operators, which act on the normed space Y and with the values in normed space V.

Lemma 1.5. *Let V be a real normed space and let $\mathbf{C}$ be a convex cone with nonempty interior in a real normed space Y. If a function $A\colon \mathbf{C} \to V$ is additive and continuous then there exists exactly one linear and continuous extention $\overline{A}\colon Y \to V$ of a function A such that $||\overline{A}||_{L(Y,V)} \leqslant M_0 ||A||_{\mathcal{L}}$ (for a constant M_0 see Lemma 1.4).*

Proof. Let $y \in Y$. It is easy to observe that there exist $y_1, y_2 \in \mathbf{C}$ such that $y = y_1 - y_2$. Let us define

$$\overline{A}(y) := A(y_1) - A(y_2).$$

It is easly seen that this definition is correct and, moreover, $\overline{A}$ is linear and continuous extension of A. The uniqueness of the extension of A is obvious. To complete the proof we have to show that

$$||\overline{A}||_{L(Y,V)} \leqslant M_0 ||A||_{\mathcal{L}}. \tag{6}$$

Let $y \in Y$ and let $y = y_1 - y_2$, for $y_1, y_2 \in \mathbf{C}$; from Lemma 1.4 we get

$$||\overline{A}(y)||_V = ||A(y_1) - A(y_2)||_V \leqslant M_0||A||_{\mathcal{L}}||y_1 - y_2||_Y = M_0||A||_{\mathcal{L}}||y||_Y,$$

and hence (6) is verified. $\square$

If $(V, ||\cdot||_V)$ is a real normed linear space then by $C^1(I, V)$ we denote the space of all continuously differentiable vector-functions $\phi\colon I \to V$. Moreover, for a non-empty subset $\mathbf{C} \subseteq V$, by $C^1(I, \mathbf{C})$ we denote the set of all functions $\phi \in C^1(I, V)$ such that $\phi(I) \subseteq \mathbf{C}$. Now let $||\phi||_{\mathrm{Lip}(I,V)}$ and $||\phi||_{C^1(I,V)}$ denote the norms on the space $C^1(I, V)$, defined as follows

$$||\phi||_{\mathrm{Lip}(I,V)} := ||\phi(0)||_V + \sup_{x_1 \neq x_2} \frac{||\phi(x_1) - \phi(x_2)||_V}{|x_1 - x_2|},$$
$$||\phi||_{C^1(I,V)} := ||\phi(0)||_V + \sup_{x \in [0,1]} ||\phi'(x)||_V;$$

the second supremum is finite since ϕ is continuously differentiable and I is a compact set. The first supremum above is also finite; it follows directly from the Mean Value Theorem

$$\frac{||\phi(x_1) - \phi(x_2)||_V}{|x_1 - x_2|} \leqslant \sup_{x \in [x_1, x_2]} ||\phi'(x)||_V. \tag{7}$$

Moreover, from inequality (7) we get (cf. [13]):

$$||\phi||_{\mathrm{Lip}(I,V)} \leqslant ||\phi||_{C^1(I,V)}.$$

Now, let F be a function defined on the interval I with the values in $cc(Z)$. From many definitions of differentiability of set-valued functions we choose the definition due to Banks and Jacobs (cf. [2]); we shall say that F is π-*differentiable* at $x_0 \in I$, if the vector-function $\pi \circ F$ is differentiable at x_0 (for π see Theorem 1.3; the differentiability defined in this way does not depend on the chosen π.) We define the space $C^1(I, cc(Z))$ as follows

$$C^1(I, cc(Z)) := \{F \in cc(Z)^I : \pi \circ F \in C^1(I, V_Z)\}.$$

Let us note that if F belongs to the space $C^1(I, cc(Z))$ then $F \in C(I, cc(Z))$, i.e. F is continuous. On the space $C^1(I, cc(Z))$ the metric may be defined as follows

$$d_{C^1(I,cc(Z))}(F_1, F_2) := ||\pi \circ F_1 - \pi \circ F_2||_{C^1(I,V_Z)},$$

where $F_1, F_2 \in C^1(I, cc(Z))$.

2. Main results

Theorem 2.1. *Let Y, Z be normed linear spaces and $\mathbf{C}$ be a convex cone in Y. Assume that the Nemytskij operator N generated by $G\colon I \times \mathbf{C} \to cc(Z)$ satisfies the following conditions*

1) $N\colon C^1(I, \mathbf{C}) \to C^1(I, cc(Z))$,

2) *there exists* $L \geqslant 0$ *such that*

$$d_{C^1(I,cc(Z))}(N\phi_1, N\phi_2) \leqslant L||\phi_1 - \phi_2||_{C^1(I,Y)}, \quad \phi_1, \phi_2 \in C^1(I, \mathbf{C}). \tag{8}$$

Then there exist functions $A\colon I \times \mathbf{C} \to cc(Z), B\colon I \to cc(Z)$ *such that* B, A_y *belongs to the space* $C^1(I, cc(Z))$ *for every* $y \in \mathbf{C}$*, the function* A^x *is additive and Lipschitzian for every* $x \in I$ *and*

$$G(x, y) = A(x, y) + B(x), \quad x \in I, y \in \mathbf{C}.$$

Moreover the function $I \ni x \mapsto A^x \in \mathcal{L}(\mathbf{C}, cc(Z))$ *satisfies the Lipschitz condition with the constant* L*, i.e.*

$$d_{\mathcal{L}}(A^{x_1}, A^{x_2}) \leqslant L|x_1 - x_2| \quad x_1, x_2 \in I. \tag{9}$$

Remark 2.2. Since in the course of the proof of Theorem 2.1 Rådström embedding is applied, it is natural to ask, whether this theorem is a conclusion from its vector-valued analogue (cf. [17]). The answer, as is easy to see, is “no” – if we formulate the vector-valued analogue and try to prove Theorem 2.1 as a corollary from this analogue, then the crucial step depends on whether the difference of two vectors from the cone is another vector from this cone. Without more detailed information on these vectors we can not generally give a positive answer.

Proof. Let us note, that $G(\cdot, y) \in C^1(I, cc(Z))$. To see that let us fix $y \in \mathbf{C}$ and set $\phi_1(x) = y, x \in I$. By 1) and from the definition of N we get $N\phi_1 = g(\cdot, y) \in C^1(I, cc(Z))$. In particular G is continuous with respect to the first variable.

Now, let us take $x_1, x_2 \in I$ such that $0 \leqslant x_1 < x_2 \leqslant 1$ and let $u, v \in \mathbf{C}$. Consider two functions $\phi_1, \phi_2\colon I \to Y$ defined by

$$\phi_1(x) := y_1 + \alpha(x)[y_2 - y_1], \tag{10}$$
$$\phi_2(x) := \widetilde{y}_1 + \alpha(x)[\widetilde{y}_2 - \widetilde{y}_1], \tag{11}$$

where α is an arbitrary function from the space $C^1(I, I)$ for which equalities $\alpha(x_1) = 0, \alpha(x_2) = 1$ holds and $y_1 = \widetilde{y}_2 := (u + v)/2, y_2 := v, \widetilde{y}_1 := u$. Note that $\phi_1, \phi_2 \in C^1(I, \mathbf{C})$ and $||\phi_1 - \phi_2||_{C^1(I,Y)} = ||(v - u)/2||_Y$. Thus, from 2) we get

$$d_{C^1(I,cc(Z))}(N\phi_1, N\phi_2) \leqslant L||(v - u)/2||_Y.$$

Hence, from the definition of the metric $d_{C^1(I,cc(Z))}$ and from inequality (7), there holds

$$|x_1 - x_2|^{-1}||(\pi \circ N\phi_1 - \pi \circ N\phi_2)(x_1) - (\pi \circ N\phi_1 - \pi \circ N\phi_2)(x_2)||_{V_Z} \leqslant \\ \leqslant L||(v - u)/2||_Y.$$

Thus, from the definition of the Nemytskij operator, we get

$$||\pi(G(x_1, \frac{u+v}{2})) + \pi(G(x_2, \frac{u+v}{2})) - [\pi(G(x_1, u)) + \pi(G(x_2, v))]||_{V_Z} \leqslant \\ \leqslant L\frac{||v - u||_Y}{2}|x_1 - x_2|,$$

and since π is an isometry we infer

$$d(G(x_1, \frac{u+v}{2}) + G(x_2, \frac{u+v}{2}), G(x_1, u)) + G(x_2, v)) \leqslant L\frac{||v-u||_Y}{2}|x_1 - x_2|.$$

Now, letting $x_1, x_2 \to x$, where x is an arbitrary point of the interval I, we get (since G is continuous with respect to the first variable)

$$G(x, \frac{u+v}{2}) = \frac{1}{2}[G(x,u) + G(x,v)]$$

and from Lemma 1.2 there is

$$G(x,y) = A(x,y) + B(x), \tag{12}$$

where $A\colon I \times \mathbf{C} \to cc(Z)$, $B\colon I \to cc(Z)$ and A is additive with respect to the second variable.

To prove that $B \in C^1(I, cc(Z))$, let us note that

$$G(x,0) = A(x,0) + B(x) = \{0\} + B(x) = B(x),$$

and $G(\cdot, 0) \in C^1(I, cc(Z))$. Now we shall prove that $A_y \in C^1(I, cc(Z))$ for every $y \in \mathbf{C}$. From equalities (2) and (12) and from definition (3) we get

$$\pi(N\phi_1(x)) = \pi(G(x,y)) = \pi(A(x,y)) + \pi(B(x)),$$

where $\phi_1(x) = y$ for $x \in I$. Hence $\pi \circ A_y = \pi \circ N\phi_1 - \pi \circ B$. Since $\pi \circ N\phi_1, \pi \circ B \in C^1(I, V_Z)$, thus $\pi \circ A_y \in C^1(I, V_Z)$, which mplies that $A_y \in C^1(I, cc(Z))$ for every $y \in \mathbf{C}$.

Now we shall prove that the inequality

$$d(G(x,y), G(x,\widetilde{y})) \leqslant L||y - \widetilde{y}||_Y, \quad x \in I, y, \widetilde{y} \in \mathbf{C} \tag{13}$$

holds. Let us fix $x \in I, y, \widetilde{y} \in C$. Define $\phi_1, \phi_2\colon I \to \mathbf{C}$ as follows: $\phi_1(t) = y, \phi_2(t) = \widetilde{y}$ for $t \in I$. It is obvious that $\phi_1, \phi_2 \in C^1(I, \mathbf{C})$. Let us note that $||\phi_1 - \phi_2||_{C^1(I,Y)} = ||y - \widetilde{y}||_Y$. According to Lemma 1.1 we get

$$\begin{aligned} d(G(x,y), G(x,\widetilde{y})) &= d(G(x,y) + G(0,\widetilde{y}), G(x,\widetilde{y}) + G(0,\widetilde{y})) \leqslant \\ &\leqslant d(G(x,y) + G(0,\widetilde{y}), G(x,\widetilde{y}) + G(0,y)) + d(G(x,\widetilde{y}) + G(0,y), G(x,\widetilde{y}) + G(0,\widetilde{y})) = \\ &= d(G(0,y), G(0,\widetilde{y})) + d(G(x,y) + G(0,\widetilde{y}), G(0,y) + G(x,\widetilde{y})). \end{aligned}$$

Since π is an isometry, we get

$$\begin{aligned} d(G(0,y), G(0,\widetilde{y})) &= ||\pi((G(0,y)) - \pi(G(0,\widetilde{y})))||_{V_Z} = \\ &= ||(\pi \circ N\phi_1)(0) - (\pi \circ N\phi_2)(0)||_{V_Z} = ||(\pi \circ N\phi_1 - \pi \circ N\phi_2)(0)||_{V_Z}. \end{aligned}$$

Moreover from Mean Value Theorem we obtain

$$\begin{aligned}
&d(G(x,y)+G(0,\widetilde{y}),G(0,y)+G(x,\widetilde{y}))= \\
&\qquad =||\pi((G(x,y))+\pi((G(0,\widetilde{y}))-\pi((G(0,y))-\pi((G(x,\widetilde{y}))||_{V_Z}= \\
&\qquad =||\pi((G(x,y))-\pi((G(x,\widetilde{y}))-[\pi((G(0,y))-\pi((G(0,\widetilde{y}))]||_{V_Z}= \\
&\qquad =||\pi(N\phi_1(x))-\pi(N\phi_2(x))-[\pi(N\phi_1(0))-\pi(N\phi_2(0))]||_{V_Z}= \\
&\qquad =||\pi\circ N\phi_1(x)-\pi\circ N\phi_2(x)-[\pi\circ N\phi_1(0)-\pi\circ N\phi_2(0)]||_{V_Z}= \\
&\qquad =||(\pi\circ N\phi_1-\pi\circ N\phi_2)(x)-(\pi\circ N\phi_1-\pi\circ N\phi_2)(0)||_{V_Z}\leqslant \\
&\qquad \leqslant \sup_{t\in[0,x]}||(\pi\circ N\phi_1-\pi\circ N\phi_2)'(t)||_{V_Z}(x-0)\leqslant \\
&\qquad \leqslant \sup_{t\in I}||(\pi\circ N\phi_1-\pi\circ N\phi_2)'(t)||_{V_Z}.
\end{aligned}$$

Thus from (8) we get

$$\begin{aligned}
d(G(x,y),G(x,\widetilde{y}))&\leqslant ||(\pi\circ N\phi_1-\pi\circ N\phi_2)(0)||_{V_Z}+\sup_{t\in I}||(\pi\circ N\phi_1-\pi\circ N\phi_2)'(t)||_{V_Z}= \\
&=||(\pi\circ N\phi_1-\pi\circ N\phi_2)||_{C^1(I,V_Z)}=d_{C^1(I,cc(Z))}(N\phi_1,N\phi_2)\leqslant L||y-\widetilde{y}||_Y,
\end{aligned}$$

which completes the proof of inequality (13). Now from (12) and from Lemma 1.1 we get

$$\begin{aligned}
d(A^x(y),A^x(\widetilde{y}))=d(A(x,y),A(x,\widetilde{y}))&=d(A(x,y)+B(x),A(x,\widetilde{y})+B(x))= \\
&=d(G(x,y),G(x,\widetilde{y}))\leqslant L||y-\widetilde{y}||_Y,
\end{aligned}$$

and we conclude that A^x is Lipschitzian.

Finally, we shall prove that (9) holds. Let $z,w\in\mathbf{C}$ and let ϕ_1,ϕ_2 be given by (10) and (11), where $y_1=\widetilde{y}_2=z+w, y_2=2z+w, \widetilde{y}_1=w$. Then $(\phi_1-\phi_2)(x)=z$ for $x\in I$. Hence, from (8) and from Mean Value Theorem we obtain

$$||\pi(G(x_1,y_1)+G(x_2,\widetilde{y}_2))-\pi(G(x_1,\widetilde{y}_1)+G(x_2,y_2))||_{V_Z}\leqslant L||y_1-y_2||_Y|x_1-x_2|.$$

Thus

$$d(G(x_1,z+w)+G(x_2,z+w),G(x_1,w)+G(x_2,2z+w))\leqslant L||z||_Y|x_1-x_2|.$$

From (12) and from additivity of function A^x for $x\in I$ we get

$$d(A(x_1,z),A(x_2,z))\leqslant L||z||_Y|x_1-x_2|.$$

Thus

$$d_{\mathcal{L}}(A^{x_1},A^{x_2})=\sup_{z\in\mathbf{C}\setminus\{0\}}\frac{d(A^{x_1}(z),A^{x_2}(z))}{||z||_Y}\leqslant L|x_1-x_2|,$$

which completes the proof. □

Remark 2.3. It is possible to formulate this theorem in a stronger form, assuming only uniform continuity of N, instead of satisfying the Lipschitz condition – see Theorem 2.6 and its proof.

Theorem 2.4. *Let Y and Z be normed linear spaces, let $\mathbf{C}$ be a convex cone with nonempty interior in the space Y and let A, B be given functions such that $A\colon I \times \mathbf{C} \to cc(Z)$ and $B\colon I \to cc(Z)$. Assume that A_y, B belong to the space $C^1(I, cc(Z))$ for $y \in \mathbf{C}$ and A^x belongs to the space $\mathcal{L}(\mathbf{C}, cc(Z))$ for $x \in I$. Moreover, let the function $I \ni x \mapsto A^x \in \mathcal{L}(\mathbf{C}, cc(Z))$ satisfies the Lipschitz condition, i.e. there exists a constant $L \geqslant 0$ such that*

$$d_{\mathcal{L}}(A^{x_1}, A^{x_2}) \leqslant L|x_1 - x_2|, \quad x_1, x_2 \in I. \tag{14}$$

If we define the function $G\colon I \times \mathbf{C} \to cc(Z)$ in the following way

$$G(x, y) = A(x, y) + B(x), \quad x \in I, y \in \mathbf{C},$$

then the Nemytski operator N generated by G maps the set $C^1(I, \mathbf{C})$ into the space $C^1(I, cc(Z))$ and satisfies the Lipschitz condition, i.e., there exists a constant $L' \geqslant 0$ such that

$$d_{C^1(I,cc(Z))}(N\phi_1, N\phi_2) \leqslant L'||\phi_1 - \phi_2||_{C^1(I,Y)}, \quad \phi_1, \phi_2 \in C^1(I, \mathbf{C}). \tag{15}$$

Proof. Let $x_1, x_2 \in I$. Since $\pi \circ A^{x_1}, \pi \circ A^{x_2}$ are additive and continuous, so is $\pi \circ A^{x_1} - \pi \circ A^{x_2}$. We shall prove now that the following inequality holds

$$||\pi \circ A^{x_1} - \pi \circ A^{x_2}||_{\mathcal{L}} \leqslant L|x_1 - x_2|. \tag{16}$$

Let $y \in \mathbf{C}$. From (4) and (14) we get

$$d(A^{x_1}(y), A^{x_2}(y)) \leqslant L||y||_Y|x_1 - x_2|,$$

whence, as π is an isometry

$$||[\pi \circ A^{x_1} - \pi \circ A^{x_2}](y)||_{V_Z} \leqslant L||y||_Y|x_1 - x_2|.$$

Thus, according to (1) and (5), (16) holds. Now let $x \in I$. By Lemma 1.5 there is exactly one linear, continuous function $\overline{\pi \circ A^x}\colon Y \to V_Z$ such that

$$\overline{\pi \circ A^x}(y) = \pi \circ A^x(y) \quad \text{for} \quad y \in \mathbf{C}.$$

We shall prove now that the function

$$I \ni x \mapsto \overline{\pi \circ A^x} \in L(Y, V_Z) \tag{17}$$

satisfies the Lipschitz condition. Let $x_1, x_2 \in I$. Obviously

$$\overline{\pi \circ A^{x_1} - \pi \circ A^{x_2}} = \overline{\pi \circ A^{x_1}} - \overline{\pi \circ A^{x_2}}.$$

Hence, from Lemma 1.5 and (6), we infer

$$\begin{aligned}||\overline{\pi \circ A^{x_1}} - \overline{\pi \circ A^{x_2}}||_{L(Y,V_Z)} = ||\overline{\pi \circ A^{x_1} - \pi \circ A^{x_2}}||_{L(Y,V_Z)} \leqslant \\ \leqslant M_0||\pi \circ A^{x_1} - \pi \circ A^{x_2}||_{\mathcal{L}} \leqslant M_0 L|x_1 - x_2|,\end{aligned}$$

and thus we get

$$||\overline{\pi \circ A^{x_1}} - \overline{\pi \circ A^{x_2}}||_{L(Y,V_Z)} \leqslant M_0 L|x_1 - x_2|. \tag{18}$$

Now we shall prove that the following inequality

$$||(\pi \circ A_{y_1})'(x) - (\pi \circ A_{y_2})'(x)||_{V_Z} \leqslant M_0 L||y_1 - y_2||_Y \tag{19}$$

holds for $x \in I$ and $y_1, y_2 \in \mathbf{C}$. Let $h \in \mathbb{R}, h \neq 0$ and let $x + h \in I$. From inequality (18) we get

$$\begin{aligned}
&||(\pi \circ A_{y_1})(x+h) - (\pi \circ A_{y_1})(x) - [(\pi \circ A_{y_2})(x+h) - (\pi \circ A_{y_2})(x)]||_{V_Z} = \\
&= ||\pi(A(x+h, y_1)) - \pi(A(x+h, y_2)) - [\pi(A(x, y_1)) - \pi(A(x, y_2))]||_{V_Z} = \\
&= ||\overline{\pi \circ A^{x+h}}(y_1) - \overline{\pi \circ A^{x+h}}(y_2) - [\overline{\pi \circ A^{x}}(y_1) - \overline{\pi \circ A^{x}}(y_2)]||_{V_Z} = \\
&= ||[\overline{\pi \circ A^{x+h}} - \overline{\pi \circ A^{x}}](y_1 - y_2)||_{V_Z} \leqslant M_0 L|h|||y_1 - y_2||_Y,
\end{aligned}$$

whence

$$\begin{aligned}
&\Big|\Big|\frac{(\pi \circ A_{y_1})(x+h) - (\pi \circ A_{y_1})(x)}{h} - \frac{(\pi \circ A_{y_2})(x+h) - (\pi \circ A_{y_2})(x)}{h}\Big|\Big|_{V_Z} \leqslant \\
&\leqslant M_0 L||y_1 - y_2||_Y.
\end{aligned}$$

Letting $t \to 0$ we get (19).

Now, we shall prove that the derivative of the function $\pi \circ N\phi \colon I \to V_Z$ at any point $x_0 \in I$ is given by the following formula

$$(\pi \circ N\phi)'(x_0) = (\pi \circ B)'(x_0) + (\pi \circ A_{\phi(x_0)})'(x_0) + \overline{\pi \circ A^{x_0}}\,(\phi'(x_0)). \tag{20}$$

Indeed, applaying in turn the definition of the derivative, the triangle inequality and equalities (2) and (3) we get

$$\begin{aligned}
&||(\pi \circ N\phi)(x_0 + h) - (\pi \circ N\phi)(x_0) - h(\pi \circ B)'(x_0) - \\
&\quad - h(\pi \circ A_{\phi(x_0)})'(x_0) - h[\overline{\pi \circ A^{x_0}}(\phi'(x_0))]||_{V_Z} = \\
&= ||\pi(A(x_0 + h, \phi(x_0 + h)) + B(x_0 + h)) - \pi(A(x_0, \phi(x_0)) + B(x_0)) - \\
&\quad - h(\pi \circ B)'(x_0) - h(\pi \circ A_{\phi(x_0)})'(x_0) - h[\overline{\pi \circ A^{x_0}}(\phi'(x_0))]||_{V_Z} \leqslant \\
&\leqslant ||(\pi \circ B)(x_0 + h) - (\pi \circ B)(x_0) - h(\pi \circ B)'(x_0)||_{V_Z} + || - h[\overline{\pi \circ A^{x_0}}(\phi'(x_0))] + \\
&\quad + \pi(A(x_0 + h, \phi(x_0 + h))) - \pi(A(x_0, \phi(x_0))) - h(\pi \circ A_{\phi(x_0)})'(x_0)||_{V_Z} \leqslant \\
&\leqslant o(h) + ||\pi(A(x_0 + h, \phi(x_0))) - \pi(A(x_0, \phi(x_0))) - h(\pi \circ A_{\phi(x_0)})'(x_0)||_{V_Z} + \\
&\quad + ||\pi(A(x_0 + h, \phi(x_0 + h))) - \pi(A(x_0 + h, \phi(x_0))) - h[\overline{\pi \circ A^{x_0}}(\phi'(x_0))]||_{V_Z} \leqslant \\
&\leqslant o(h) + ||(\pi \circ A_{\phi(x_0)})(x_0 + h)) - (\pi \circ A_{\phi(x_0)})(x_0)) - h(\pi \circ A_{\phi(x_0)})'(x_0)||_{V_Z} + \\
&\quad + ||\pi(A(x_0, \phi(x_0 + h)) - \pi(A(x_0, \phi(x_0)) - h[\overline{\pi \circ A^{x_0}}(\phi'(x_0))] + \\
&\quad + \pi(A(x_0 + h, \phi(x_0 + h)) - \pi(A(x_0, \phi(x_0 + h)) - \\
&\quad - \pi(A(x_0 + h, \phi(x_0)) + \pi(A(x_0, \phi(x_0))||_{V_Z} \leqslant \\
&\leqslant o(h) + o(h) + ||(\pi \circ A^{x_0})(\phi(x_0 + h)) - (\pi \circ A^{x_0})(\phi(x_0)) - h\,\overline{\pi \circ A^{x_0}}(\phi'(x_0))||_{V_Z} +
\end{aligned}$$

$$
\begin{aligned}
&+ ||(\pi \circ A^{x_0+h})(\phi(x_0+h)) - (\pi \circ A^{x_0+h})(\phi(x_0)) - \\
&\qquad - (\pi \circ A^{x_0})(\phi(x_0+h)) + (\pi \circ A^{x_0})(\phi(x_0))||_{V_Z} \leqslant \\
&\leqslant o(h) + ||\overline{\pi \circ A^{x_0}}(\phi(x_0+h)) - \overline{\pi \circ A^{x_0}}(\phi(x_0)) - \overline{\pi \circ A^{x_0}}(\phi'(x_0)(h))||_{V_Z} + \\
&\qquad + ||\overline{\pi \circ A^{x_0+h}}(\phi(x_0+h)) - \overline{\pi \circ A^{x_0+h}}(\phi(x_0)) - \\
&\qquad - \overline{\pi \circ A^{x_0}}(\phi(x_0+h)) + \overline{\pi \circ A^{x_0}}(\phi(x_0))||_{V_Z} \leqslant \\
&\leqslant o(h) + ||\overline{\pi \circ A^{x_0}}\,[\phi(x_0+h) - \phi(x_0) - \phi'(x_0)(h)]||_{V_Z} + \\
&\qquad + ||\overline{\pi \circ A^{x_0+h}}\,[\phi(x_0+h) - \phi(x_0)] + \overline{\pi \circ A^{x_0}}\,[\phi(x_0+h) - \phi(x_0)]||_{V_Z} \leqslant \\
&\leqslant o(h) + ||\overline{\pi \circ A^{x_0}}\,[\phi(x_0+h) - \phi(x_0) - \phi'(x_0)(h)]||_{V_Z} + \\
&\qquad + ||[\,\overline{\pi \circ A^{x_0+h}} - \overline{\pi \circ A^{x_0}}\,](\phi(x_0+h) - \phi(x_0))||_{V_Z} \leqslant \\
&\leqslant o(h) + ||\overline{\pi \circ A^{x_0}}||_{L(Y,V_Z)}\, ||[\phi(x_0+h) - \phi(x_0) - \phi'(x_0)(h)]||_Y + \\
&\qquad + ||[\,\overline{\pi \circ A^{x_0+h}} - \overline{\pi \circ A^{x_0}}\,]||_{L(Y,V_Z)}\, ||(\phi(x_0+h) - \phi(x_0))||_Y, \qquad (21)
\end{aligned}
$$

where $o(h)$ is the Landau symbol.

Let us note that there exists a constant $M \geqslant 0$ such that $||\overline{\pi \circ A^x}||_{L(Y,V_Z)} \leqslant M$ for all $x \in I$, since the function (17) is continuous and its domain is a compact set. Now let us write $\phi(x_0+h) = \phi(x_0) + h\phi'(x_0) + |h|f(h)$, where $||f(h)||_Y \to 0$ for $h \to 0$. Thus, from inequality (18), (21) is less than or equal to

$$o(h) + Mo(h) + M_0L[\phi'(x_0) + f(h)]\,|h|^2 = o(h),$$

which completes the proof of (20). Moreover, (20) implies that N maps the set $C^1(I,\mathbf{C})$ into the space $C^1(I, cc(Z))$.

Now we shall prove that there exists a constant $L' \geqslant 0$ such that (15) holds. Let ϕ_1 and ϕ_2 belong to the space $C^1(I,\mathbf{C})$. From the definition of the metric $d_{C^1(I,cc(Z))}$ and from the definition of the norm $||\cdot||_{C^1(I,V_Z)}$ we get

$$
\begin{aligned}
&d_{C^1(I,cc(Z))}(N\phi_1, N\phi_2) = ||\pi \circ N\phi_1 - \pi \circ N\phi_2||_{C^1(I,V_Z)} = \\
&\quad = ||(\pi \circ N\phi_1)(0) - (\pi \circ N\phi_2)(0)||_{V_Z} + \sup_{x\in I} ||(\pi \circ N\phi_1)'(x) - (\pi \circ N\phi_2)'(x)||_{V_Z}.
\end{aligned}
$$

According to (19) and (20) we infer

$$
\begin{aligned}
&||(\pi \circ N\phi_1)'(x_0) - (\pi \circ N\phi_2)'(x_0)||_{V_Z} \leqslant \\
&\qquad \leqslant ||(\pi \circ A_{\phi_1(x_0)})'(x_0) - (\pi \circ A_{\phi_2(x_0)})'(x_0)||_{V_Z} + \\
&\qquad\quad + ||\overline{\pi \circ A^{x_0}}\,(\phi_1'(x_0)) - \overline{\pi \circ A^{x_0}}\,(\phi_2'(x_0))||_{V_Z} \leqslant \\
&\qquad \leqslant M_0L||\phi_1(x_0) - \phi_2(x_0)||_Y + ||\overline{\pi \circ A^{x_0}}[\phi_1'(x_0) - \phi_2'(x_0)]||_{V_Z}.
\end{aligned}
$$

It is easily seen that $||\phi_1(x_0) - \phi_2(x_0)||_Y \leqslant ||\phi_1 - \phi_2||_{C^1(I,Y)}$, thus

$$
\begin{aligned}
&\sup_{x\in I} ||(\pi \circ N\phi_1)'(x) - (\pi \circ N\phi_2)'(x)||_{V_Z} \leqslant \\
&\qquad\qquad \leqslant M_0L||\phi_1 - \phi_2||_{C^1(I,Y)} + M \sup_{x\in I} ||(\phi_1 - \phi_2)'(x)||_Y.
\end{aligned}
$$

Moreover,

$$\begin{aligned}
&||(\pi \circ N\phi_1)(0) - (\pi \circ N\phi_2)(0)||_{V_Z} = \\
&\qquad = ||\pi(A(0,\phi_1(0)) + B(0)) - \pi(A(0,\phi_2(0)) + B(0))||_{V_Z} = \\
&= ||\overline{\pi \circ A^0}(\phi_1(0)) - \overline{\pi \circ A^0}(\phi_2(0))||_{V_Z} \leqslant ||\overline{\pi \circ A^0}||_{L(Y,V_Z)}\, ||(\phi_1 - \phi_2)(0)||_Y \leqslant \\
&\qquad\qquad \leqslant M||(\phi_1 - \phi_2)(0)||_Y.
\end{aligned}$$

It follows that

$$d_{C^1(I,cc(Z))}(N\phi_1, N\phi_2) \leqslant L'||\phi_1 - \phi_2||_{C^1(I,Y)},$$

where $L' := M + M_0 L \, (\geqslant 0)$. □

Now lets turn our attention to absolutely continuous functions with values in the space $cc(\mathbb{R})$. For brevity, in the following we shall write $\mathcal{K}$ instead of $cc(\mathbb{R})$. Thus $\mathcal{K}$ consists of all non-empty, compact intervals (including degenerate ones) in $\mathbb{R}$. For $I, J \in \mathcal{K}$ and $\alpha \in \mathbb{R}$ we define $I + J := \{x + x' : x \in I, x' \in J\}$ and $\alpha I := \{\alpha x : x \in I\}$. It is clear that if $[a,b], [c,d] \in \mathcal{K}$ and $\alpha \geqslant 0$, then

$$\begin{aligned}
[a,b] + [c,d] &= [a+c, b+d], \\
\alpha[a,b] &= [\alpha a, \alpha b].
\end{aligned}$$

Of course we have

$$d([a,b],[c,d]) = \max\{|a-c|, |b-d|\}.$$

Now consider the norm $||\cdot||$ in $\mathbb{R}^2$, defined by

$$||(x,y)|| := \max\{|x|,|y|\}, \quad (x,y) \in \mathbb{R}^2,$$

and the map

$$\pi : \mathcal{K} \ni [a,b] \mapsto (a,b) \in \mathbb{R}^2. \tag{22}$$

It is clear that the following relations

$$\begin{aligned}
\pi([a,b] + [c,d]) &= \pi([a,b]) + \pi([c,d]), \\
\pi(\alpha[a,b]) &= \alpha\pi([a,b]) \quad \text{for} \quad \alpha \geqslant 0, \\
d([a,b],[c,d]) &= ||\pi([a,b]) - \pi([c,d])||, \\
\mathbb{R}^2 &= \pi(\mathcal{K}) - \pi(\mathcal{K})
\end{aligned}$$

hold. Thus, $\mathcal{K}$ can be embedded into the (of course, complete and reflexive) space $\mathbb{R}^2$, endowed with the maximum norm (cf. [11]). Moreover, a space into which $\mathcal{K}$ is embedded in the above fashion is unique up to isometrical isomorphism – see Theorem 1.3.

A function $\phi\colon I \to Y$ is said to be *absolutely continuous* (cf. [3], p. 15), if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any positive integer N and a disjoint family of intervals $(\alpha_1,\beta_1), (\alpha_2,\beta_2), \dots, (\alpha_N,\beta_N)$ in I whose lengths satisfy $\sum_{i=1}^N (\beta_i - \alpha_i) < \delta$, the inequality $\sum_{i=1}^N ||\phi(\beta_i) - \phi(\alpha_i)||_Y < \epsilon$ holds true. We define absolute continuity of a set-valued function $F\colon I \to \mathcal{K}$ in a similar manner; instead of the distance generated by the norm $||\cdot||_Y$, we consider the Hausdorff distance in $\mathcal{K}$. We denote the space of all absolutely continuous functions defined on the interval I and with the values in the space Y by $AC(I,Y)$. Moreover, for a subset $\mathbf{C} \subseteq Y$, by $AC(I,\mathbf{C})$ we denote the set of all functions $\phi \in AC(I,Y)$ such that $\phi(I) \subseteq \mathbf{C}$.

Lemma 2.5 ([4], p. 44, Theorem 3.4)**.** *Let Y be a reflexive Banach space and let a function $\phi\colon I \to Y$ be absolutely continuous. Then ϕ is differentiable a.e. on the interval I (with respect to the Lebesgue measure), ϕ' is integrable in the sense of Bochner and*

$$\phi(x) - \phi(a) = \int_{[a,x]} \phi'(t)dt \quad \text{for} \quad a, x \in I.$$

Now, let $(Y, ||\cdot||_Y)$ be a reflexive Banach space and let us define the norm $||\cdot||_{AC(I,Y)}$ in the space $AC(I,Y)$ in the following way

$$||\phi||_{AC(I,Y)} := ||\phi(0)||_Y + \int_{[0,1]} ||\phi'(t)||_Y dt; \tag{23}$$

since ϕ' is integrable the integral in (23) is finite. It is easy to see, that the following relation holds

$$F \in AC(I,\mathcal{K}) \Leftrightarrow \pi \circ F \in AC(I,\mathbb{R}^2),$$

where π is the Rådström embedding (22). Since the function π is invertible, thus the formula

$$d_{AC(I,\mathcal{K})}(F_1,F_2) := ||\pi \circ F_1 - \pi \circ F_2||_{AC(I,\mathbb{R}^2)}, \quad F_1, F_2 \in AC(I,\mathcal{K}), \tag{24}$$

defines a metric in $AC(I,\mathcal{K})$.

Now we shall formulate the following set-valued analogue of the J. Matkowski's result (cf. [17], Theorem 1 and [18]).

Theorem 2.6. *Let Y be a reflexive Banach space and let $\mathbf{C}$ be a convex cone in Y. Assume that the Nemytskij operator N generated by $G\colon I \times \mathbf{C} \to \mathcal{K}$ maps the set $AC(I,\mathbf{C})$ into the space $AC(I,\mathcal{K})$ and is uniformly continuous. Then there exist functions $A\colon I \to \mathcal{L}(\mathbf{C},\mathcal{K}), B\colon I \to \mathcal{K}$ such that B and $A(\cdot)\,y$ belong to the space $AC(I,\mathcal{K})$ for every $y \in \mathbf{C}$, the function $A(x)(\cdot)$ is uniformly continuous for every $x \in I$ and*

$$G(x,y) = A(x)y + B(x), \quad x \in I, y \in \mathbf{C}.$$

Also, if we assume that there exists $L \geqslant 0$, such that

$$d_{AC(I,\mathcal{K})}(N\phi_1, N\phi_2) \leqslant L||\phi_1 - \phi_2||_{AC(I,Y)}, \quad \phi_1, \phi_2 \in AC(I,\mathbf{C}), \tag{25}$$

then the function $A(x)$ satisfies the Lipschitz condition with the constant L for every $x \in I$.

Proof. Let us note, that $G(\cdot,y) \in AC(I,\mathcal{K})$. To see that let us fix $y \in \mathbf{C}$ and set $\phi_1(x) = y, x \in I$. From the definition of N and assumption, that N maps the set $AC(I,\mathbf{C})$ into the space $AC(I,\mathcal{K})$, we get $N\phi_1 = G(\cdot,y) \in AC(I,\mathcal{K})$. In particular G is continuous with respect to the first variable.

Now, let us take $x_1, x_2, ..., x_{2n} \in I$ such that $0 \leqslant x_1 < x_2 < ... < x_{2n} \leqslant 1$ and let $u, v \in \mathbf{C}$. Moreover, let us define $y_1 = \overline{y}_2 := (u+v)/2, y_2 := v, \overline{y}_1 := u$. Consider a function $\phi_1\colon I \to Y$ defined by

$$\phi_1(x) = \begin{cases} y_1 & \text{for } x \in [0, x_1], \\ y_1 + \frac{x - x_{2i-1}}{x_{2i} - x_{2i-1}}[\overline{y}_1 - y_1] & \text{for } x \in [x_{2i-1}, x_{2i}], \\ \overline{y}_1 + \frac{x - x_{2i}}{x_{2i+1} - x_{2i}}[y_1 - \overline{y}_1] & \text{for } x \in [x_{2i}, x_{2i+1}], \\ \overline{y}_1 & \text{for } x \in [x_{2n}, 1], \end{cases} \tag{26}$$

Moreover, let us define a function $\phi_2 \colon I \to Y$, by putting $y_2, \overline{y}_2$ instead of $y_1, \overline{y}_1$, respectively, in definition (34). It is not difficult to check that $\phi_1, \phi_2 \in AC(I, \mathbf{C})$ and $||\phi_1 - \phi_2||_{AC(I,Y)} = ||(v-u)/2||_Y$. Now, let us note (cf. [18]), that there exists a function $\gamma\colon [0, +\infty) \to [0, +\infty)$, which is continuous at 0, satisfies the condition $\gamma(0) = 0$, and, moreover, for which the inequality

$$d_{AC(I,\mathcal{K})}(N\phi_1, N\phi_2) \leqslant \gamma(\, ||\phi_1 - \phi_2||_{AC(I,Y)}), \quad \phi_1, \phi_2 \in AC(I, \mathbf{C})$$

holds; in fact, we can take

$$\gamma(t) := \sup\{d_{AC(I,\mathcal{K})}(N\phi_1, N\phi_2) : \phi_1, \phi_2 \in AC(I, C), \quad ||\phi_1 - \phi_2||_{AC(I,Y)} \leqslant t\}.$$

It is not difficult to show, that uniform continuity of N implies, that the values of γ are finite, $\gamma(0) = 0$ and γ is continuous at 0. Thus, from Lemma 2.5, (23) and (24) we get

$$\begin{aligned} \gamma(\, ||(v-u)/2||_Y) &\geqslant d_{AC(I,\mathcal{K})}(N\phi_1, N\phi_2) \geqslant \int_I ||(\pi \circ N\phi_1 - \pi \circ N\phi_2)'(t)||dt \geqslant \\ &\geqslant \sum_{k=1}^{2n-1} || \int_{[x_k, x_{k+1}]} (\pi \circ N\phi_1 - \pi \circ N\phi_2)'(t)dt|| = \\ &= \sum_{k=1}^{2n-1} ||\pi \circ N\phi_1(x_{k+1}) - \pi \circ N\phi_2(x_{k+1}) - \pi \circ N\phi_1(x_k) + \pi \circ N\phi_2(x_k)|| = \\ &= \sum_{k=1}^{2n-1} ||\pi(G(x_{k+1}, \phi_1(x_{k+1}))) - \pi(G(x_{k+1}, \phi_2(x_{k+1}))) - \\ &\quad - \pi(G(x_k, \phi_1(x_k))) + \pi(G(x_k, \phi_2(x_k)))|| = \\ &= \sum_{k=1}^{n} ||\pi(G(x_{2k}, \overline{y}_2)) + \pi(G(x_{2k-1}, y_1)) - \pi(G(x_{2k}, \overline{y}_1)) - \pi(G(x_{2k-1}, y_2))|| + \\ &+ \sum_{k=1}^{n-1} ||\pi(G(x_{2k}, \overline{y}_2)) + \pi(G(x_{2k+1}, y_1)) - \pi(G(x_{2k}, \overline{y}_1)) - \pi(G(x_{2k+1}, y_2))||. \end{aligned}$$

Let x be an arbitrary point of interval I and let $x_k \to x$ for all $k = 1, 2, \ldots, 2n-1$. From the continuity of G with respect to the first variable and from the definitions of $y_1, y_2, \overline{y}_1, \overline{y}_2$, we infer that

$$\begin{aligned} (2n-1)||\pi(G(x,(u+v)/2) + \pi(G(x,(u+v)/2) - \pi(G(x,u)) - \pi(G(x,v))|| &\leqslant \\ \leqslant \gamma(\, ||(v-u)/2||_Y). \end{aligned}$$

According to the fact that π is an isometry, we obtain

$$d\Big(G\Big(x,\frac{u+v}{2}\Big)+G\Big(x,\frac{u+v}{2}\Big),G(x,u)+G(x,v)\Big)\leqslant\frac{1}{2n-1}\gamma\Big(\Big|\Big|\frac{v-u}{2}\Big|\Big|_Y\Big),$$

and the above inequality holds for each positive integer n. Hence we get

$$G(x,(u+v)/2)=[G(x,u)+G(x,v)]/2$$

for arbitrary x from I and u,v from $\mathbf{C}$, i.e. G satisfies the Jensen equation with respect to the second variable. By virtue of Lemma 1.2 we obtain

$$G(x,y)=A(x)y+B(x), \tag{27}$$

where $B\colon I\to\mathcal{K}$ and $A(x)\colon\mathbf{C}\to\mathcal{K}$, for every $x\in I$.

To prove that $B\in AC(I,\mathcal{K})$, let us note that

$$G(x,0)=A(x)0+B(x)=\{0\}+B(x)=B(x).$$

Moreover, $G(\cdot,0)=N\phi$, where the function $\phi\colon I\to\mathbf{C}$ is given by $\phi(x)=0$ for $x\in I$. Since N takes its values in the space $AC(I,\mathcal{K})$, B is absolutely continuous.

Now we shall prove that $A(\cdot)y\in AC(I,\mathcal{K})$ for $y\in\mathbf{C}$. Fix $y\in\mathbf{C}$. Let us consider a function $\phi\colon I\to\mathbf{C}$ given by $\phi(x)=y$ for $x\in I$. From definition (3), equality (27) and from additivity of π we get

$$\pi(N\phi(x))=\pi(G(x,y))=\pi(A(x)y)+\pi(B(x)).$$

Hence $\pi\circ A(\cdot)y=\pi\circ N\phi-\pi\circ B$. Moreover $\pi\circ N\phi,\pi\circ B\in AC(I,\mathbb{R}^2)$. Thus $\pi\circ A(\cdot)y\in AC(I,\mathbb{R}^2)$, which implies that $A(\cdot)y\in AC(I,\mathcal{K})$.

Now we shall prove that the inequality

$$d(G(x,y_1),G(x,y_2))\leqslant\gamma(\,||y_1-y_2||_Y),\quad x\in I,y_1,y_2\in\mathbf{C} \tag{28}$$

holds. Let us fix $x\in I,y_1,y_2\in\mathbf{C}$. Define $\phi_1,\phi_2\colon I\to\mathbf{C}$ as follows: $\phi_1(t)=y_1,\phi_2(t)=y_2$ for $t\in I$. It is obvious that $\phi_1,\phi_2\in AC(I,\mathbf{C})$. Let us note that $||\phi_1-\phi_2||_{AC(I,Y)}=||y_1-y_2||_Y$. Since π is an isometry, from triangle inequality we get

$$\begin{aligned}d(G(x,y_1),G(x,y_2))&=||\pi(G(x,y_1))-\pi(G(x,y_2))||\leqslant\\&\leqslant||\pi(G(0,y_1))-\pi(G(0,y_2))||+\\&\quad+||\pi(G(x,y_1))-\pi(G(x,y_2))-[\pi(G(0,y_1))-\pi(G(0,y_2))]||.\end{aligned}$$

Moreover, since

$$d(G(0,y_1),G(0,y_2))=||(\pi\circ N\phi_1-\pi\circ N\phi_2)(0)||$$

and

$$\begin{aligned}&||\pi(G(x,y_1))-\pi(G(x,y_2))-[\pi(G(0,y_1))-\pi(G(0,y_2))]||=\\&\qquad=||(\pi\circ N\phi_1-\pi\circ N\phi_2)(x)-(\pi\circ N\phi_1-\pi\circ N\phi_2)(0)||=\\&\qquad=\Big|\Big|\int_{[0,x]}(\pi\circ N\phi_1-\pi\circ N\phi_2)'(t)dt\Big|\Big|\leqslant\int_I||(\pi\circ N\phi_1-\pi\circ N\phi_2)'(t)||dt,\end{aligned}$$

we have

$$
\begin{aligned}
&d(G(x,y_1),G(x,y_2)) \leqslant \\
&\qquad \leqslant ||(\pi\circ N\phi_1-\pi\circ N\phi_2)(0)||+\int_I ||(\pi\circ N\phi_1-\pi\circ N\phi_2)'(t)||dt= \\
&\qquad = ||(\pi\circ N\phi_1-\pi\circ N\phi_2)||_{AC(I,\mathbb{R}^2)} = d_{AC(I,\mathcal{K})}(N\phi_1,N\phi_2)\leqslant \\
&\qquad\qquad \leqslant \gamma(\,||\phi_1-\phi_2||_{AC(I,Y)})\leqslant \gamma(\,||y_1-y_2||_Y),
\end{aligned}
$$

which completes the proof of inequality (28). Now from (28) and from Lemma 1.1 we get

$$
d(A(x)y_1,A(x)y_2)=d(G(x,y_1),G(x,y_2))\leqslant \gamma(\,||y_1-y_2||_Y).
$$

Thus $A(x)$ is uniformly continuous for $x\in I$, and hence A is a map from I into $\mathcal{L}(\mathbf{C},\mathcal{K})$. Moreover, if the operator N satisfies (25), then in an analogous way we can show, that the function $A(x)$ satisfies the Lipschitz condition with the constant L for all x from the interval I, which completes the proof. $\square$

In the following theorem we shall give sufficient conditions for the Nemytskij operator to be lipschitzian. Let us note, that the reflexivity of the space Y implies the reflexivity of the space $L(Y,\mathbb{R}^2)$ (cf. [8]).

Theorem 2.7. *Let Y be a reflexive Banach space, and let $\mathbf{C}$ be a convex cone with nonempty interior in the space Y. Moreover, let A and B be given functions such that $A\colon I\to\mathcal{L}(\mathbf{C},\mathcal{K})$ and $B\colon I\to\mathcal{K}$. Assume, that the functions B and $I\ni x\mapsto A(x)\in\mathcal{L}(\mathbf{C},\mathcal{K})$ are absolutely continuous (the set $\mathcal{L}(\mathbf{C},\mathcal{K})$ is endowed with the metric (4)). If $G\colon I\times\mathbf{C}\to\mathcal{K}$ is of the form*

$$
G(x,y)=A(x)y+B(x),\quad x\in I, y\in\mathbf{C},
$$

then the Nemytskij operator N generated by G maps the set $AC(I,\mathbf{C})$ into the space $AC(I,\mathcal{K})$ and satisfies the Lipschitz condition, i.e., there exists a constant $L\geqslant 0$ such that

$$
d_{AC(I,\mathcal{K})}(N\phi_1,N\phi_2)\leqslant L||\phi_1-\phi_2||_{AC(I,Y)},\quad \phi_1,\phi_2\in AC(I,\mathbf{C}).
$$

Proof. Let $x\in I$. Since the function $\pi\circ A(x)\colon\mathbf{C}\to\mathbb{R}^2$ is additive and continuous, from Lemma 1.5 we get, that there is exactly one linear and continuous function $\overline{\pi\circ A(x)}\colon Y\to\mathbb{R}^2$ such that

$$
\overline{\pi\circ A(x)}(y)=[\pi\circ A(x)](y)\quad\text{for}\quad y\in\mathbf{C}.
$$

We divide the proof into five steps.

1. We shall prove that the functions

$$
I\ni x\mapsto \overline{\pi\circ A(x)}\in L(Y,\mathbb{R}^2) \tag{29}
$$

and $A(\cdot)y$ for $y\in\mathbf{C}$ are absolutely continuous. Let $0\leqslant\alpha_1<\beta_1\leqslant\alpha_2<\beta_2\leqslant ...\leqslant \alpha_N<\beta_N\leqslant 1$. Since $\pi\circ A(\beta_i),\pi\circ A(\alpha_i)$ are additive and continuous for $i=1,2,...,N$, the function $\pi\circ A(\beta_i)-\pi\circ A(\alpha_i)$ is also additive and continuous. For $y\in\mathbf{C}\setminus\{0\}$ from (4) we get

$$d(A(\beta_i)y, A(\alpha_i)y) \leqslant ||y||_Y d_{\mathcal{L}(\mathbf{C},\mathcal{K})}(A(\beta_i), A(\alpha_i)),$$

which implies, that the function $A(\cdot)y$ is absolutely continuous for $y \in \mathbf{C}$. Moreover,

$$\frac{1}{||y||_Y} ||[\pi \circ A(\beta_i) - \pi \circ A(\alpha_i)](y)|| \leqslant d_{\mathcal{L}(\mathbf{C},\mathcal{K})}(A(\beta_i), A(\alpha_i)).$$

Thus

$$||\pi \circ A(\beta_i) - \pi \circ A(\alpha_i)||_{\mathcal{L}(\mathbf{C},\mathbb{R}^2)} \leqslant d_{\mathcal{L}(\mathbf{C},\mathcal{K})}(A(\beta_i), A(\alpha_i)).$$

Obviously

$$\overline{\pi \circ A(\beta_i) - \pi \circ A(\alpha_i)} = \overline{\pi \circ A(\beta_i)} - \overline{\pi \circ A(\alpha_i)}.$$

Hence, from Lemma 1.5 we infer

$$||\overline{\pi \circ A(\beta_i)} - \overline{\pi \circ A(\alpha_i)}||_{L(Y,\mathbb{R}^2)} \leqslant M_0 ||\pi \circ A(\beta_i) - \pi \circ A(\alpha_i)||_{\mathcal{L}(\mathbf{C},\mathcal{K})}$$

and thus we get

$$\sum_{i=1}^{N} ||\overline{\pi \circ A(\beta_i)} - \overline{\pi \circ A(\alpha_i)}||_{L(Y,\mathbb{R}^2)} \leqslant M_0 \sum_{i=1}^{N} d_{\mathcal{L}(\mathbf{C},\mathcal{K})}(A(\beta_i), A(\alpha_i)).$$

Since the function $I \ni x \mapsto A(x) \in \mathcal{L}(I, \mathcal{K})$ is absolutely continuous, so is the function $I \ni x \mapsto \overline{\pi \circ A(x)} \in L(Y, \mathbb{R}^2)$.

2. Now we shall prove that N maps the set $AC(I, \mathbf{C})$ into the space $AC(I, \mathcal{K})$.
Let us note that there exists a constant K such that

$$||\overline{\pi \circ A(x)}||_{L(Y,\mathbb{R}^2)} \leqslant K, \quad x \in I. \tag{30}$$

In fact the function (29) is continuous (even absolutely continuous) and its domain is a compact set. Moreover, let us note, that inequality (30) implies that the function $A(x)$ satisfies the Lipschitz condition with the constant K.

Let $\phi \in AC(I, \mathbf{C})$ and consider the function $I \ni x \mapsto A(x)\phi(x) \in \mathcal{K}$. Moreover, let $0 \leqslant \alpha_1 < \beta_1 \leqslant \alpha_2 < \beta_2 \leqslant ... \leqslant \alpha_N < \beta_N \leqslant 1$. Since $A(x)$ satisfies the Lipschitz condition with the constant K for every x from I, we get

$$\begin{aligned}
&d(A(\alpha_i)\phi(\alpha_i), A(\beta_i)\phi(\beta_i)) \leqslant \\
&\qquad \leqslant d(A(\alpha_i)\phi(\alpha_i), A(\alpha_i)\phi(\beta_i)) + d(A(\alpha_i)\phi(\beta_i), A(\beta_i)\phi(\beta_i)) \leqslant \\
&\qquad \leqslant K||\phi(\alpha_i) - \phi(\beta_i)||_Y + ||[\pi \circ A(\alpha_i)]\phi(\beta_i) - [\pi \circ A(\beta_i)]\phi(\beta_i)|| \leqslant \\
&\qquad \leqslant K||\phi(\alpha_i) - \phi(\beta_i)||_Y + ||[\overline{\pi \circ A(\alpha_i)} - \overline{\pi \circ A(\beta_i)}]\phi(\beta_i)|| \leqslant \\
&\qquad \leqslant K||\phi(\alpha_i) - \phi(\beta_i)||_Y + \max_{t \in I} ||\phi(t)||_Y ||[\overline{\pi \circ A(\alpha_i)} - \overline{\pi \circ A(\beta_i)}]||_{L(Y,\mathbb{R}^2)},
\end{aligned}$$

whence

$$\sum_{i=1}^{N} d(A(\alpha_i)\phi(\alpha_i), A(\beta_i)\phi(\beta_i)) \leqslant$$
$$\leqslant K \sum_{i=1}^{N} ||\phi(\alpha_i) - \phi(\beta_i)||_Y + \max_{t \in I} ||\phi(t)||_Y \sum_{i=1}^{N} ||[\overline{\pi \circ A(\alpha_i)} - \overline{\pi \circ A(\beta_i)}]||_{L(Y,\mathbb{R}^2)}.$$

Since the functions $I \ni x \mapsto \phi(x) \in \mathbf{C}$ and $I \ni x \mapsto \overline{\pi \circ A(x)} \in L(Y, \mathbb{R}^2)$ are absolutely continuous, so is the function $I \ni x \mapsto A(x)\phi(x) \in \mathcal{K}$. Therefore, the function $N\phi$, given by $N\phi(x) = A(x)\phi(x) + B(x)$, is absolutely continuous.

3. Now, let $x \in (0,1)$ and assume that the derivatives $\phi'(x), (\pi \circ B)'(x)$ and $(\pi \circ N\phi)'(x)$ exist (since the functions $\phi, \pi \circ B, \pi \circ N\phi$ are absolutely continuous, their derivatives exist a.e. on the interval I). We shall prove that the following formula holds

$$(\pi \circ [A(\cdot)\phi(x)])'(x) = (\pi \circ N\phi)'(x) - (\pi \circ B)'(x) - \overline{\pi \circ A(x)}\,(\phi'(x)). \tag{31}$$

Since ϕ is differentiable at x we can write $\phi(x+h) = \phi(x) + h\phi'(x) + hf(h)$, where f is a function defined on the neighbourhood of 0 in $\mathbb{R}$ and with the values in the space Y, such that $\lim_{h \to 0} f(h) = 0$. For $h \in \mathbb{R}$ with sufficiently small $|h|$ we have

$$\begin{aligned}
&(\pi \circ N\phi)(x+h) - (\pi \circ N\phi)(x) = \\
&\quad = \pi(A(x+h)\phi(x+h)) + \pi(B(x+h)) - \pi(A(x)\phi(x)) - \pi(B(x)) = \\
&\qquad = \pi(A(x+h)\phi(x)) - \pi(A(x)\phi(x)) + h\,\overline{\pi \circ A(x+h)}\phi'(x) + \\
&\qquad\qquad + h\,\overline{\pi \circ A(x+h)}f(h) + \pi(B(x+h)) - \pi(B(x));
\end{aligned}$$

whence

$$\begin{aligned}
&(1/h)[\pi(A(x+h)\phi(x)) - \pi(A(x)\phi(x))] = \\
&= (1/h)[(\pi \circ N\phi)(x+h) - (\pi \circ N\phi)(x)] - (1/h)[(\pi \circ B)(x+h) - (\pi \circ B)(x)] - \\
&\qquad - \overline{\pi \circ A(x+h)}\phi'(x) - \overline{\pi \circ A(x+h)}f(h),
\end{aligned}$$

which implies that (31) holds.

4. Now, let $x \in I, y_1, y_2 \in \mathbf{C}$, and let us assume that the derivatives of the functions $\pi \circ [A(\cdot)y_1], \pi \circ [A(\cdot)y_2], \overline{\pi \circ A(\cdot)}$ exist at the point x. It is easily seen that the inequality

$$||(\pi \circ [A(\cdot)y_1])'(x) - (\pi \circ [A(\cdot)y_2])'(x)|| \leqslant ||\left[\,\overline{\pi \circ A(\cdot)}\,\right]'(x)||_{L(Y,\mathbb{R}^2)}||y_1 - y_2||_Y \tag{32}$$

holds for $x \in I, y_1, y_2 \in \mathbf{C}$.

5. Now we shall prove that the operator N is lipschitzian. Let ϕ_1 and ϕ_2 belong to the set $AC(I, \mathbf{C})$. From (23) and (24) we get

$$d_{AC(I,\mathcal{K})}(N\phi_1, N\phi_2) = ||\pi \circ N\phi_1 - \pi \circ N\phi_2||_{AC(I,\mathbb{R}^2)} =$$
$$= ||(\pi \circ N\phi_1)(0) - (\pi \circ N\phi_2)(0)|| + \int_{[0,1]} ||(\pi \circ N\phi_1)'(x) - (\pi \circ N\phi_2)'(x)||dx.$$

According to (31) and (32) we infer

$$||(\pi \circ N\phi_1)'(x) - (\pi \circ N\phi_2)'(x)|| \leqslant$$
$$\leqslant ||(\pi \circ [A(\cdot)\phi_1(x)])'(x) - (\pi \circ [A(\cdot)\phi_2(x)])'(x)|| + ||\overline{\pi \circ A(x)}\phi_1'(x) - \overline{\pi \circ A(x)}\phi_2'(x)|| \leqslant$$
$$\leqslant ||\left[\, \overline{\pi \circ A(\cdot)}\,\right]'(x)||_{L(Y,\mathbb{R}^2)}||\phi_1(x) - \phi_2(x)||_Y +$$
$$+ \ ||\overline{\pi \circ A(x)}||_{L(Y,\mathbb{R}^2)}||\phi_1'(x) - \phi_2'(x)||.$$

It is easy to see that $||\phi_1(x) - \phi_2(x)||_Y \leqslant ||\phi_1 - \phi_2||_{AC(I,Y)}$ for $x \in I$; thus

$$\int_{[0,1]} ||(\pi \circ N\phi_1)'(x) - (\pi \circ N\phi_2)'(x)||\, dx \leqslant$$
$$\leqslant ||\phi_1 - \phi_2||_{AC(I,Y)} \int_{[0,1]} ||\left[\, \overline{\pi \circ A(\cdot)}\,\right]'(x)||_{L(Y,\mathbb{R}^2)}\, dx + K \int_{[0,1]} ||(\phi_1 - \phi_2)'(x)||_Y\, dx.$$

Moreover,

$$||(\pi \circ N\phi_1)(0) - (\pi \circ N\phi_2)(0)|| =$$
$$= ||\pi(A(0)\phi_1(0) + B(0)) - \pi(A(0)\phi_2(0) + B(0))|| =$$
$$= ||\overline{\pi \circ A(0)}(\phi_1(0)) - \overline{\pi \circ A(0)}(\phi_2(0))|| \leqslant$$
$$\leqslant ||\overline{\pi \circ A(0)}||_{L(Y,\mathbb{R}^2)}\, ||(\phi_1 - \phi_2)(0)||_Y \leqslant K||(\phi_1 - \phi_2)(0)||_Y$$

and finally

$$d_{AC(I,\mathcal{K})}(N\phi_1, N\phi_2) \leqslant L||\phi_1 - \phi_2||_{AC(I,Y)},$$

where $L = K + \int_{[0,1]} ||\left[\, \overline{\pi \circ A(\cdot)}\,\right]'(x)||_{L(Y,\mathbb{R}^2)}\, dx\ (\geqslant 0)$. □

Remark 2.8. It is a natural reaction to try to generalize the method of proofs of Theorems 2.6 and 2.7 to the more general situations. Thus we are led to the question, whether the Hausdorff completion of the Rådström space V_Z is reflexive. The positive answer to the simplest case of the space $cc(\mathbb{R})$ is given above. However, even in the case of $cc(\mathbb{R}^n)$ for $n > 1$, this problem is still unresolved.

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